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On a ring of formal pseudo-differential operators

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We begin with a definition of the higher local fields arising in algebraic geometry as purely local objects attached to algebraic varieties of arbitrary dimension.

Let K and k be some fields. We say that K has a structure of a n-*dimensional local field* with the (last) residue field k if either $n = 0$ and $K = k$ or $n \geq 1$ and K is the quotient field of a complete discrete valuation ring $\mathcal{O}_K$ whose residue field $\bar{K}$ is a local field of dimension $n - 1$ with the last residue field k.

A typical example is the field of iterated Laurent power series

$$K = k((x_1)) \dots ((x_n))$$

with the following local structure:

$$\mathcal{O}_K = k((x_1)) \dots ((x_{n-1}))[[x_n]]$$

$$\bar{K} = k((x_1)) \dots ((x_{n-1}))$$

$$\dots\dots\dots$$

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If the characteristics of all the residue fields are equal, then the field K must be of this form (for other examples and a more general classification theorem see [4, ch.2]).

This construction has first appeared in algebraic number theory for $n = 1$ and it was later used in the theory of algebraic curves over an arbitrary field. It can be defined for varieties and schemes of arbitrary dimension (see a survey [4]) and has numerous applications to the problems of algebraic geometry, both arithmetical and geometrical.

From this viewpoint, it would be reasonable to restrict ourselves by the commutative fields in the definition given above. Nevertheless, in the class field theory we meet the rings of this kind but the non-commutative ones. We mean the skew-fields which are finite-dimensional over their center K. Thus the K will be a (commutative) local field and the skew-fields will represent the elements of the Brauer group of the field K (see [11]).

The main purpose of this note is to point out onto another class of non-commutative local fields arising in the theory of differential equations and to show that these skew-fields hold many features of the commutative fields. We define a skew-field P of formal pseudo-differential operators in n variables which will be an $2n$-dimensional local field. In contrast with the previous examples, it will be infinite-dimensional over its center. The main properties of the skew-field P and of it's "order" E are given in §§1 and 2 [1].

We note that for the case of one variable a wider class of twisted pseudo-differential operators was considered in general algebra [2] but there the purposes and motivations are purely algebraic and have no relations neither to algebraic geometry, nor to differential equations.

It would be very interesting to get a classification of non-commutative local fields at least for the case of equal characteristics of the residue fields [2]. The commutative case is exhausted by the example of the Laurent power series. This classification should contain as a particular case a description of the rings of pseudo-differential operators and an explicit construction of skew-fields from the Brauer group.

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[1] There are other approaches to the rings of pseudo-differential operators in n variables, see [6], [7] [3]. We have used some arguments from [7]. The work [3] became known to the author after the present text had been written.

[2] Several interesting results along this line were recently proved by A. B. Zheglov, see his papers in Uspekhi Matem. Nauk, **54**(1999), N 4, pp. 177-178 and Izvestija RAN (to appear)

Another interesting problem is to find a generalization of the construction of commutative local field from a chain of subvarieties of an algebraic variety [4]. It would be reasonable to define a class of non-commutative schemes and to extract the non-commutative local fields by an appropriate localization process.

The paper is completed by §§3 and 4, where we show that the Kadomtsev-Petviashvili hierarchy (the dynamical system defined on the space $\partial + E_-$ for the case $n = 1$) can be extended to the space P^n. The usual properties of the KP hierarchy will be preserved under the new circumstances, in particular, the existence of infinitely many conservation laws, the zero curvature presentation as the Zakharov-Shabat equations, etc. A connection of this generalization of the KP hierarchy and of some natural Poisson structures on the space P^n is discussed in §4.

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1 Skew-field P and ring E

Let A be an associative not necessary commutative ring and let $d : A \to A$ be its derivation. Let us first introduce the ring $A((\partial^{-1}))$ of formal pseudo-differential operators with coefficients from A as a left A-module of all formal expressions

$$L = \sum_{i > -\infty}^{n} a_i \partial^i, \ a_i \in A.$$

Then a multiplication can be defined according to the Leibnitz rule:

$$(\sum_i a_i \partial^i)(\sum_j b_j \partial^j) = \sum_{i,j;k \geq 0} \binom{i}{k} a_i d^k(b_j) \partial^{i+j-k}.$$

Here we put

$$\binom{i}{k} = \frac{i(i-1)\dots(i-k+1)}{k(k-1)\dots 1}, \text{ if } k > 0, \ \binom{i}{0} = 1.$$

Particularly, for $a \in A$:

$$[\partial, a] = \partial a - a\partial = d(a),$$

(Heisenberg commutative relation),

$$[\partial^{-1}, a] = \partial^{-1}a - a\partial^{-1} = -d(a)\partial^{-2} + d^2(a)\partial^{-3} - \dots .$$

It can be checked that $A((\partial^{-1}))$ will be again an associative ring. For the case of a ring A of functions, this ring was introduced by I. Schur [14]. Later the ring $A((\partial^{-1}))$ was many times rediscovered and studied (see [5], [13], [9], [8]). The case of a non-commutative ring A was considered at the beginning of the paper [8].

We can iterate the constuction and this gives an opportunity to consider n variables and to introduce the following rings.

Definition 1. Let k be a field and let $x_1, \dots x_n$ be some commuting variables. We put

$$P = k((x_1)) \dots ((x_n))((\partial_1^{-1})) \dots ((\partial_n^{-1})),$$
$$E = k[[x_1, \dots x_n]]((\partial_1^{-1})) \dots ((\partial_n^{-1})).$$

Then E is a subring in P. Here $k((x_1)) \dots ((x_n))$ is the ring of iterated Laurent power series and $k[[x_1, \dots x_n]]$ is the ring of Taylor power series. In the first case the order of variables $x_1, \dots x_n$ is significant.

If $L = \sum_{i \le m} a_i \partial_n^i$ and $a_m \ne 0$, then $m = \mathrm{ord}(L)$ will be called the *order* of the operator L. The function $\mathrm{ord}(.)$ defines a decreasing *filtration* $P_. : \dots \subset P_{-1} \subset P_0 \subset \dots$ of vector subspaces $P_i = \{L \in P \ : \mathrm{ord}(L) \le i\} \subset P$.

We have the decomposition of P and, accordingly of E, in a direct sum of subspaces

$$P = P_+ + P_-,$$
$$E = E_+ + E_-, \ E_\pm = E \cap P_\pm,$$

where $P_- = \{L \in P : \mathrm{ord}(L) < 0\}$ and P_+ consists of the operators containing only ≥ 0 powers of ∂_n.

Let us define the *highest term* of an operator L by induction on n. If $L = \sum_{i \le m} a_i \partial_n^i$ and $\mathrm{ord}(L) = m$ then we put

$$\text{highest term}(L) = (\text{highest term}(a_m))\partial_n^m.$$

The highest term has the following form $f\partial_1^{m_1} \dots \partial_n^{m_n}, \ f \in k((x_1)) \dots ((x_n))$, $f \ne 0$ and one can define

$$\nu(L) = (m_1, \dots m_n) \in \mathbf{Z}^n.$$

Proposition 1 . *The rings P and E have the following properties:*

i) P is an associative skew-field; an operator $L \in E$ is invertible in the ring $E \Leftrightarrow$ the coefficient f in the highest term of L is invertible in the ring $k[[x_1, \dots x_n]]$

ii)

$$ord(LM) = ord(L) + ord(M),$$

$$ord(L+M) \leq max(ord(L), ord(M))$$

iii)

$$\nu(LM) = \nu(L) + \nu(M),$$

$$\nu(L+M) \leq max(\nu(L), \nu(M))$$

for the lexicographical order in $\mathbf{Z}^n$.

iv) Let $\nu(L)$ be divisible by $m \in \mathbf{N}$ such that

$(m, char(k) = 1$. If the coefficient f in the highest term of the operator $L \in P$ (or $\in E$) is a m-th power in $k((x_1))\dots((x_n))$(or correspondingly in $k[[x_1, \dots x_n]]$), then there exists a unique, up to multiplication by a m-th root of unity, operator $M \in P$ (or $\in E$) such that $L = M^m$.

PROOF. For the case $n = 1$, these claims are well known (see [14], [9], [7]). The general case can be done by a simple induction over n. We only note that the proof of associativity for the ring P can be given exactly as for $n = 1$ ([9][ch. III, §11]) using lemma 1.

It is important to note that inequality

$$\mathrm{ord}([L, M]) \geq \mathrm{ord}(L) + \mathrm{ord}(M) - 1,$$

known for $n = 1$, is in general wrong for $n > 1$ (see, for example, equality (5) §2).

Thus P_0 is a discrete valuation ring in P with residue field

$$k((x_1))\dots((x_n))((\partial_1^{-1}))\dots((\partial_{n-1}^{-1}))$$

and we can introduce a structure of a $2n$-dimensional local field on P by an induction (considering also the skew-fields with a non-necessarily equal number of variables and derivations)

Now let $V = k((x_1))\dots((x_n))((z_1))\dots((z_n))$. This is a field. *The symbol* of an operator $L = \sum a_{i_1\dots i_n}\partial_1^{i_1}\dots\partial_n^{i_n}$ is the series $\sigma(L) = \sum a_{i_1\dots i_n} z_1^{i_1}\dots z_n^{i_n}$. For $i, j = 1, \dots n$ there are linear maps

$$d_i,\ D_j : V \to V, \text{where } d_i = \partial/\partial x_i;\ D_j = \partial/\partial z_j.$$

Let us introduce several notations which are standard for the theory of differential operators in n variables. We denote by α, β, γ the elements from $\mathbf{Z}^n$. Thus we will use the following abbreviations for operators

$$\partial^\alpha = \partial_1^{\alpha_1}\dots\partial_n^{\alpha_n},\ d^\alpha = d_1^{\alpha_1}\dots d_n^{\alpha_n},\ D^\alpha = D_1^{\alpha_1}\dots D_n^{\alpha_n},$$

variables $z^\alpha = z_1^{\alpha_1}\dots z_n^{\alpha_n}$ and coefficients

$$\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1}\dots\binom{\alpha_n}{\beta_n},\ \alpha! = \alpha_1!\dots\alpha_n!,$$

where $\alpha = (\alpha_1, \dots \alpha_n)$. We also write $\alpha \geq 0$ if all the $\alpha_i \geq 0$.

Lemma 1 . *Let $F,\ G \in V$. Put*

$$F * G = \sum_{\alpha \geq 0} \frac{1}{\alpha!} D^\alpha F d^\alpha G.$$

For operators $L,\ M \in P$ we then have

$$\sigma(LM) = \sigma(L) * \sigma(M).$$

Proof. It is sufficient to check the lemma for $L = a\partial^\alpha,\ M = b\partial^\beta$. Then $\sigma(L) = az^\alpha,\ \sigma(M) = bz^\beta$ and we get

$$\sigma(L) * \sigma(M) = \sum_{\gamma \geq 0} \frac{1}{\gamma!} D^\gamma \sigma(L) d^\gamma \sigma(M) = \sum_{\gamma \geq 0} \frac{1}{\gamma!} a D^\gamma(z^\alpha)\partial^\gamma(b) z^\beta =$$

$$\sum_{\gamma \geq 0} \binom{\alpha}{\gamma} a\partial^\gamma(b) z^{\alpha+\beta-\gamma} = \sigma(LM).$$

Now we go to the constructions in the ring P related to duality.

Definition 2. Let $L \in P$ $\ L = \sum a_{i_1\dots i_n; j_1\dots j_n} x_1^{i_1}\dots x_n^{i_n}\partial_1^{j_1}\dots\partial_n^{j_n}$. Then the *residue* of the operator L is defined as

$$\mathrm{res}_P(L) = a_{-1\dots-1} \in k.$$

If $L, M \in P$, then we put

$$< L, M >= \mathrm{res}_P(LM).$$

Proposition 2 . *The residue res_P and the pairing $< ., >$ have the following properties:*

i) res_P is a linear form on P and

$$res_P(L) = res_V(\sigma(L))$$

ii) for any $L,\ M \in P$

$$res_P([L, M]) = 0.$$

iii) the pairing $< ., . >$ is a nondegenerate bilinear form on P.

Proof. i) This claim is obvious. ii) Let $K = k((x_1))\dots((x_n))$. Then $\mathrm{res}_P(L) = \mathrm{res}_{K/k}(\mathrm{res}_{P/K}(L))$. Let us show that if $L \in [P,\ P]$, then

$$\mathrm{res}_{P/K}(L) \in d_1(K) + \ldots + d_n(K).$$

This gives what we need. By lemma 1 and property i), it is enough to check that we have

$$F * G - G * F \in d_1(V) + \ldots + d_n(V) + D_1(V) + \ldots + D_n(V) \qquad (1)$$

in the field V of symbols. Remembering the definition of $*$, we can restrict ourselves by considerating the term $D^\alpha(F)d^\alpha(G) - D^\alpha(G)d^\alpha(F)$ from the commutator in the left hand side of (1). Let us write $f \sim g$ if $f - g$ belongs to the right hand side of (1). Since d_i and D_j are derivations, we have

$$D^\alpha(F)d^\alpha(G) - D^\alpha(G)d^\alpha(F) = D^\alpha(F)d^\alpha(G) - d^\alpha(F)D^\alpha(G) \sim$$

$$FD^\alpha d^\alpha(G) - Fd^\alpha D^\alpha(G) = 0,$$

because $D^\alpha d^\alpha = d^\alpha D^\alpha$. The proof given here is an adaptation to our situation of the arguments from [7][IV.50].

iii) follows from

$$< x^\alpha \partial^\beta,\ x^{\alpha'} \partial^{\beta'} >=$$

$$\begin{cases} 1 & \text{if } \alpha+\alpha'=\beta+\beta'=(-1,\dots,-1) \\ 0 & \text{if for some } i\ \alpha_i+\alpha_i'+1<0 \text{ or } \beta_i+\beta_i'+1<0. \end{cases}$$

REMARK 1. The statement iii) shows that the skew-field P is auto-dual. Thus it has the most important property of the n-dimensional local fields.

REMARK 2. Although we consider in this paper a purely formal situation let us mention that the constructions and results given above can be mostly extended to the case of ring $\mathcal{R} = R((\partial_1^{-1}))\dots((\partial_n^{-1}))$ where R is the ring of C^∞ functions on $\mathbf{R}^n$ rapidly decreasing at infinity. Proposition 1 can be reformulated word by word. Concerning proposition 2, one has to consider the linear form

$$tr : \mathcal{R} \to \mathbf{R},$$

$$tr(\sum a_{i_1\dots i_n}\partial_1^{i_1}\dots\partial_n^{i_n}) = \int\limits_{\mathbf{R}^n} a_{-1,\dots -1}dx_1\dots dx_n$$

instead of the operation res_P.

2 The conjugacy theorems

We assume that the characteristic of the field k is equal to 0. Denote by $(<m)$ an arbitrary operator of an order $<m$.

Theorem 1 . *Let $L_1 \in \partial_1 + E_-, \dots L_n \in \partial_n + E_-$ or $L_1 \in \partial_1 + \mathcal{R}_-, \dots L_n \in \partial_n + \mathcal{R}_-$. Then the following conditions are equivalent:*

i) for any i,j $[L_i, L_j] = 0$

ii) There exists an operator $S \in 1 + E_-$ (or $\in 1 + \mathcal{R}_-$) such that

$$L_1 = S^{-1}\partial_1 S, \dots L_n = S^{-1}\partial_n S.$$

If S, S' are two operators from ii) then

$$S'S^{-1} \in 1 + k((\partial_1^{-1}))\dots((\partial_n^{-1})) \cap E_-$$

in the first case and

$$S'S^{-1} \in 1 + \mathbf{R}((\partial_1^{-1}))\dots((\partial_n^{-1})) \cap \mathcal{R}_-$$

in the second case.

PROOF can be done by the same reasoning in both cases and, to be concrete, we take the ring E. Denote by R the ring $k[[x_1, \dots x_n]]$ and by E' the ring $R((\partial_1^{-1}))\dots((\partial_{n-1}^{-1}))$.

Now let

$$L_i = \partial_i + M_i = \partial_i + a_i\partial_n^m + (< m),\ m < 0,\ a_i \in E'. \tag{2}$$

A straightforward computation gives

$$[L_i, L_j] = (\frac{\partial a_j}{\partial x_i} - \frac{\partial a_i}{\partial x_j})\partial_n^m + (< m), \tag{3}$$

where we have extended the derivations $\frac{\partial}{\partial x_i}$ from the ring R to E as above.

If

$$S = 1 - P,\ P = b\partial_n^m,\ b \in E', \tag{4}$$

then it is easy to check that

$$[\partial_i, P] = \frac{\partial b}{\partial x_i}\partial_n^m. \tag{5}$$

Now we are ready to prove the theorem using subsequent approximations in powers of ∂_n.

Let the operators L_i satisfy the condition i) and have the form as in (2). We will look for an S of the form (4). We have

$$S^{-1}L_iS = (1 + P + P^2 + \dots)L_i(1 - P) =$$

$$L_i - [L_i, P] - P[L_i, P] - P^2[L_i, P] - \dots =$$

$$\partial_i + M_i - [\partial_i, P] - [M_i, P] - P[\partial_i, P] - \dots.$$

According to proposition 1, all the terms, except the first three , belong to $E_{<m}$. Hence, we get

$$S^{-1}L_iS = \partial_i + (a_i - \frac{\partial b}{\partial x_i})\partial_n^m + (< m).$$

Since the commutators from (3) are all equal to zero, it implies that there exists $b \in E'$ such that

$$a_i = \frac{\partial b}{\partial x_i}, \text{for all } i = 1, \dots n.$$

It means that if we choose S with such b, then the second term in our decomposition is equal to zero and thus the operators L_i are conjugate to ∂_i up to terms of the next order.

The operators S obtained in this way step by step can be multiplied inside the group $1 + E_-$ and the result is a solution of our problem. The inverse implication $ii) \Rightarrow i)$ is obvious.

In order to get the second statement of the theorem it is enough to note that the conditions imply

$$[S'S^{-1}, \partial_i] = 0, \text{for all } i = 1, \dots n.$$

and, again applying the formula (5) by induction, we get that all coefficients of the operator $S'S^{-1}$ are constant.

This completes our proof.

Now denote by $Z(M)$ the ring of operators commuting with every operator from M.

Corollary 1 . *Let $L_1 \in \partial_1 + E_-, \dots L_n \in \partial_n + E_-$ and the condition i) from theorem 1 is satisfied. Then the ring $Z(L_1, \dots L_n) \subset E$ is commutative and is equal to $k((L_1^{-1}))\dots((L_n^{-1}))$.*

Proof. By theorem 1, we can assume that $L_i = \partial_i$ for all i. It is obvious that $Z(L_1, \dots L_n) \supset k((\partial_1^{-1}))\dots((\partial_n^{-1}))$. The equality of these rings will follow from the next refinement of the formula (5).

Let

$$P = b\partial_n^m + (< m), \ b \in E', \ m \in \mathbf{Z}.$$

Then

$$[\partial_i, P] = \frac{\partial b}{\partial x_i}\partial_n^m + (< m).$$

Proposition 3 . *The center of the ring P is equal to k.*

Proof is done using the formula just given (it is also valid in P). Namely, if $L \in Z(P)$, then L commutes with all the ∂_i and, therefore, it has constant coefficients. Because L does also commute with all x_i, we get that $L \in k$.

For the case $n = 1$, theorem 1 was proved by M. Sato [13], [8]. In this form, the result cannot be extended to the skew-field P. Already for $n = 1$, there are non-trivial obstacles to the conjugacy of operators, which do not

exist for the case of regular coefficients. In Sato's paper there is an extension of theorem 1 for $n = 1$, which uses so-called quasi-regular operators (both for L and S). Here we give another simple result of this kind but also for $n = 1$.

Theorem 2 . *Let L, M be operators of orders $\neq 0$ from the ring P of one variable. Assume that they have the form $\partial^m + (< m-1)$. Then the following conditions are equivalent:*

i) There is $S \in 1 + P_-$ such that $M = S^{-1}LS$

ii) $ord(L) = ord(M) =: k$ and for all $m \in \frac{1}{k}\mathbf{Z}$

$$res_P(L^m) = res_P(M^m).$$

Proof. The implication $i) \Rightarrow ii)$ follows from propositions 1 and 2 of section 1. To get the back implication, we reduce our claim to the case $k = 1$, taking the root (proposition 1, iv)), . Next, we proceed as in the proof of theorem 1 (we only put $x_1 = x$ and $\partial_1 = \partial$).

Let

$$L = M + a\partial^m + (< m), \; m \leq -1.$$

Then

$$L^{-m} = M^{-m} + a\partial^m M^{-m-1} + Ma\partial^m M^{-m-2} + \ldots + M^{-m-1}a\partial^m + (< m) =$$

$$ma\partial^{-1} + M^{-m} + (< m),$$

because $M = \partial + (< 0)$. The conditions of the theorem show that

$$\mathrm{res}_K(a) = 0 \text{ in the field } K = k((x)).$$

Therefore, there exists an element $b \in K$ such that $\partial b/\partial x = a$.

Looking for S as $1 + b\partial^m$, we get as above

$$S^{-1}LS = M + (a - \partial b/\partial x)\partial^m + (< m) = M + (< m).$$

It is interesting to compare this result with the classification of the conjugate elements in division algebras D which are finite-dimensional over their center K. We have

Theorem 3 . *Let $X, Y \in D^*$. Then the following conditions are equivalent:*
i) There exists $U \in D^*$ such that

$$X = U^{-1} Y U$$

ii) $[K(X) : K] = [K(Y) : K]$ *and for all* $i \in \mathbf{N}$

$$Tr_{D/K}(X^i) = Tr_{D/K}(Y^i).$$

PROOF. The conditions in ii) mean that the elements X and Y have the same minimal polynomials over K and thus are conjugate in an algebraic closure of the field K. By the Skolem-Noether theorem [1][ch. VIII], there exists an element U of the algebra D which realize this conjugation inside the D.

REMARK 3. We immediately see that the conjugacy conditions in theorems 2 and 3 have something common. The valuation ord(.) corresponds to the degree [:] and the residue res_P corresponds to the trace $\mathrm{Tr}_{D/K}$. To understand an origin of this analogy, one has to remember the adelic construction of the direct image for symbols and differentials for the morphisms of schemes of relative dimension k[4]. The degree [:] is a dimension and ord(.) is a valuation and they are connected by a boundary map from K-theory. This map from K_1 to K_0 will appear when one defines the direct image of the K-sheaves for $k = 1$. In its turn, the trace and residue is used for a definition of the direct image of differential forms, correspondingly, for $k = 0$ and $k \geq 1$. (The case of the map of a surface onto a curve was considered in [10]).

At last, we note that the relative dimension (in the sense of local fields) of the skew-field D over its center K is equal to zero and this dimension for the skew-field P over its center k is greater then zero.

3 A generalization of the KP hierarchy

The vector space P and its power P^n can be considered as infinite-dimensional varieties over the field k. Particularly, in every point $L = (L_1, \ldots, L_n) \in P^n$, the tangent space T_L is again P^n. The variety P^n contains a subvariety

$$P' = \{L \in P^n : \text{for all } i,\ j\ [L_i, L_j] = 0\}.$$

Let us denote by m (and also by k and l) the multi-index $(m_1, \ldots, m_n)$ with nonnegative m_i and introduce on P^n the vector field V^m.

Definition 3.

$$V_L^m = ([(L_1^{m_1} \cdots L_n^{m_n})_+, L_1], \ldots, [(L_1^{m_1} \cdots L_n^{m_n})_+, L_n]) \in T_L.$$

This field defines a dynamical system on P^n which has the following form

$$\frac{\partial L}{\partial t_m} = V_L^m. \tag{6}$$

If we want to speak about the solutions of the system (6) in our purely formal setting, we have to consider the operators L as belonging to the extended phase space $P^n \otimes k[[\ldots, t_m, \ldots]]$. This space contains an infinitely many "times" t_m. We will also use an abbreviation $L^m = L_1^{m_1} \cdots L_n^{m_n}$.

Proposition 4 . *Let $L \in P'$. Then we have*

i) if L satisfies to the system (6) then L also satisfies to the system

$$\frac{\partial (L^m)_+}{\partial t_k} - \frac{\partial (L^k)_+}{\partial t_m} = [(L^k)_+, (L^m)_+] \tag{7}$$

ii) if $S \in 1 + P_-$ satisfies to the equation

$$\frac{\partial S}{\partial t_m} = -(S \partial_1^{m_1} \cdots \partial_n^{m_n} S^{-1})_- S, \tag{8}$$

then $L = (S\partial_1 S^{-1}, \ldots, S\partial_n S^{-1})$ satisfies to the system (6).

Proof. i). Since both parts in (6) are derivations, we have

$$\frac{\partial L^m}{\partial t_k} = [(L^k)_+, L^m]$$

and simultaneously

$$\frac{\partial (L^m)_+}{\partial t_k} = [(L^k)_+, L^m]_+.$$

Thus

$$\frac{\partial (L^m)_+}{\partial t_k} - \frac{\partial (L^k)_+}{\partial t_m} - [(L^k)_+, (L^m)_+] =$$

$$([(L^k)_+, L^m] - [(L^m)_+, L^k] - [(L^k)_+, (L^m)_+])_+ =$$

$$([(L^m)_+ - L^m, (L^k)_+ - L^k])_+ = [(L^m)_-, (L^k)_-]_+ = 0,$$

because for all $L \in P'$ we have $[L^m, L^k] = 0$.

ii). Since

$$\frac{\partial S}{\partial t_m} = -(L_1^{m_1} \cdots L_n^{m_n})_- S = -(L^m)_- S,$$

we get

$$\frac{\partial L_i}{\partial t_m} = \frac{\partial S}{\partial t_m} \partial_i S^{-1} - S \partial_i S^{-1} \frac{\partial S}{\partial t_m} S^{-1} =$$
$$-[(L^m)_-, L_i] = [(L_1^{m_1} \cdots L_n^{m_n})_+, L_i].$$

Proposition 5 . *Let $L \in P'$. Then*

i) $V_L^m \in T_{P',L}$,

ii) vector fields $V^m \mid_{P'}$ and $V^n \mid_{P'}$ commute.

Proof. i). If $L \in P'$, then the tangency condition for the vector $(X_1, \dots X_n)$ at the point L means that for all pairs i, j we have

$$[L_i + \epsilon X_i, L_j + \epsilon X_j] = 0, \text{ up to } \epsilon^2,$$

and thus $[X_i, L_j] = [X_j, L_i]$. The Jacobi identity for the Lie algebra P shows that any field of the form $([U, L_1], \dots, [U, L_n])$ will be tangent to P'.

ii). Let $L(\dots, t_m, \dots)$ be a formal solution of (6) such that $L(0) \in P'$. Then

$$\frac{\partial}{\partial t_m}(\frac{\partial}{\partial t_k} L_i) - \frac{\partial}{\partial t_k}(\frac{\partial}{\partial t_m} L_i) = \frac{\partial}{\partial t_m}[(L^k)_+, L_i] - \frac{\partial}{\partial t_k}[(L^m)_+, L_i] =$$

$$[\frac{\partial}{\partial t_m}(L^k)_+, L_i] + [(L^k)_+, \frac{\partial}{\partial t_m} L_i] - [\frac{\partial}{\partial t_k}(L^m)_+, L_i] - [(L^m)_+, \frac{\partial}{\partial t_k} L_i] =$$

$$[\frac{\partial}{\partial t_m}(L^k)_+ - \frac{\partial}{\partial t_k}(L^m)_+, L_i] + [(L^k)_+, [(L^m)_+, L_i]] - [(L^m)_+, [(L^k)_+, L_i]] =$$

$$[[(L^m)_+, (L^k)_+], L_i]] + [(L^k)_+, [(L^m)_+, L_i]] - [(L^m)_+, [(L^k)_+, L_i]] = 0.$$

Here we have used the Zakharov-Shabat equations (7) and the Jacobi identity in the Lie algebra P. The proof is completed.

Definition 4. Let $L \in P^n$. Then we put

$$H_k(L) = \mathrm{res}_P(L^k).$$

Proposition 6 . *The functions H_k are the integrals of motion for the system (6) on the manifold P^n.*

PROOF.

$$\frac{\partial}{\partial t_m} H_k = \mathrm{res}_P(\frac{\partial}{\partial t_m} L^k) =$$

$$\mathrm{res}_P((\frac{\partial}{\partial t_m} L_1^{k_1}) L_2^{k_2} \dots L_n^{k_n} + \dots + L_1^{k_1} \dots L_{n-1}^{k_{n-1}} \frac{\partial}{\partial t_m} L_n^{k_n}) =$$

$$\mathrm{res}_P([L^m, L_1^{k_1}] L_2^{k_2} \dots L_n^{k_n} + \dots + L_1^{k_1} \dots L_{n-1}^{k_{n-1}} [L^m, L_n^{k_n}]) =$$

$$\mathrm{res}_P(L^m L_1^{k_1}) \dots L_n^{k_n} - L_1^{k_1} \dots L_n^{k_n} L^m) = 0.$$

The space P^n has many submanifolds, which are invariant with respect to the system (6). One can construct them using the following result.

Lemma 2 . *Let $V(L) = L_{i_1} \cdots L_{i_k}$ be a monome in operators $L_1, \dots, L_n$ and $U \in P$. Then the vector field F such that $F_L = ([U_+, L_1], \dots, [U_+, L_n])$ is tangent to the variety $M_V = \{(L_1, \dots, L_n) \in P^n : V(L)_- = 0\}$.*

PROOF. Let $F_i = [U_+, L_i],\ i = 1, \dots, n$. The tangency condition has the following form

$$((L_{i_1} + \epsilon F_{i_1}) \cdots (L_{i_k} + \epsilon F_{i_k}))_- = 0, \text{ up to } \ \epsilon^2.$$

The left hand side is equal to

$$V(L) + \epsilon \sum_{j=1}^{k} L_{i_1} \cdots L_{i_{j-1}} F_{i_j} L_{i_{j+1}} \cdots L_{i_k} + \epsilon^2 \dots,$$

and, by application of the operation $(.)_-$, we get

$$(\sum_{j=1}^{k} L_{i_1} \cdots L_{i_{j-1}} U_+ L_{i_j} L_{i_{j+1}} \cdots L_{i_k} - \sum_{j=1}^{k} L_{i_1} \cdots L_{i_{j-1}} L_{i_j} U_+ L_{i_{j+1}} \cdots L_{i_k})_- =$$

(here all the terms, except the first two, will be cancelled)

$$(U_+ L_{i_1} \cdots L_{i_k} - L_{i_1} \cdots L_{i_k} U_+)_- = ([U_+, V])_- = [U_+, V_+]_- = 0.$$

This lemma gives an opportunity to define the different versions of the KdV hierarchy for $n > 1$.

Remark 4. All definitions and results of this section will be valid for the system (6) on the space $\mathcal{R}^n$ (instead of P^n) and with $\mathrm{tr}(L^k)$ (instead of $\mathrm{res}_P(L^k)$).

If $n = 1$, then the system (6) is the Kadomtsev-Petviashvili hierarchy extended to the whole space P (it is common to consider the system on the affine space $\partial + E_-$). Accordingly, (7) and (8) are the Zakharov-Shabat and Sato-Wilson equations. The last ones give rise to the system (6) if we restrict it to $((\partial_1, \ldots, \partial_n) + E_-^n) \cap P'$ and apply theorem 1.

4 Poisson structures on P^n and Hamiltonians H_k

We want to define a Poisson structure on the space P^n and let us first introduce the space $\mathcal{F}(P^n)$ of functionals on P^n as the vector space of polynomial functions in (any number) coefficients (close to $x_1^{i_1} \cdots x_n^{i_n} \partial_1^{j_1} \cdots \partial_n^{j_n}$) of operators $L_1, \ldots, L_n \in P$. The bilinear form $< .,. >$ on P can be naturally extended to the P^n:

$$< L,\ M >= \sum_{i=1}^{n} < L_i, M_i >,$$

where $L = (L_1, \ldots, L_n),\ M = (M_1, \ldots, M_n) \in P^n$.

If $F \in \mathcal{F}(P^n)$, then we can define the *gradient* $\nabla_F(L) \in T_L = P^n$ of the functional F at a point $L \in P^n$ by the condition

$$< M, \nabla_F(L) >= \frac{d}{d\epsilon} F(L + \epsilon M) \mid_{\epsilon=0}, \text{ for all } M \in P^n.$$

The structure of a Lie algebra on P^n gives us an opportunity to define a *Poisson bracket* $\{F, G\} \in \mathcal{F}(P^n)$ for all $F,\ G \in \mathcal{F}(P^n)$:

$$\{F, G\}(L) =< L, [\nabla_F(L), \nabla_G(L)] > .$$

The decomposition $P^n = (P^n)_+ + (P^n)_-$ gives rise to a new structure of a Lie algebra $[\ ,\ ]_R$ defined by an R-matrix (= difference of the projectors $(P^n)_+$ and $(P^n)_-$, see [12]). We have $[X, Y]_R = \frac{1}{2}[RX, Y] + \frac{1}{2}[X, RY] = [X_+, Y_+] - [X_-, Y_-]$. Thus we have introduced a new Poisson structure $\{.,.\}_R$ on P^n.

We need two simple facts.

Let $A, B, C \in P^n$. Then

$$< A, [B, C] >=< [A, B], C > . \tag{9}$$

Indeed, in any ring $A[B, C] = [A, B]C + [B, AC]$ and we have to apply proposition 2, ii).

Lemma 3 . *If $F_M \in \mathcal{F}(P^n)$ such that $F_M(L) =< L, M >$, then $\nabla_{F_M} = M$.*

PROOF immediately follows from the definition of gradient and non-degenerateness of the form $< ., . >$ on P^n.

The functions H_k from §3 belong to $\mathcal{F}(P^n)$. Let us remember that $k = (k_1, \ldots, k_n)$ and all $k_i \geq 0$.

Lemma 4 . *Let $L \in P'$. Then*

$$\nabla_{H_k}(L) = (U_1, \ldots, U_n),$$

$$\textit{where } U_i = k_i L_1^{k_1} \cdots L_{i-1}^{k_{i-1}} L_i^{k_i - 1} L_{i+1}^{k_{i+1}} \cdots L_n^{k_n}, i = 1, \ldots, n.$$

PROOF.

$$< M, \nabla_{H_k}(L) >=$$

$$\frac{d}{d\epsilon} \mathrm{res}_P((L_1 + \epsilon M_1)^{k_1} \cdots (L_n + \epsilon M_n)^{k_n}) \mid_{\epsilon=0}=$$

$$\mathrm{res}_P(\sum_{i=1}^{n} L_1^{k_1} \cdots L_{i-1}^{k_{i-1}} (\sum_{j=0}^{k_i - 1} L_i^j M_i L_i^{k_i - j - 1}) L_{i+1}^{k_{i+1}} \cdots L_n^{k_n}) =$$

$$\mathrm{res}_P(\sum_{i=1}^{n} M_i L_1^{k_1} \cdots L_{i-1}^{k_{i-1}} \sum_{j=0}^{k_i - 1} (L_i^{k_i - 1}) L_{i+1}^{k_{i+1}} \cdots L_n^{k_n}) =$$

(using proposition 2, ii) and that the L_i commute with each other)

$$\mathrm{res}_P(\sum_{i=1}^{n} M_i k_i L_1^{k_1} \cdots L_{i-1}^{k_{i-1}} L_i^{k_i - 1} L_{i+1}^{k_{i+1}} \cdots L_n^{k_n}) =$$

$$< M, U_i > .$$

Now we can compute the Poisson brackets for the functionals H_k.

Proposition 7 . *Let $L \in P'$. Then for all $k, l \quad F \in \mathcal{F}(P^n)$*

$$i)\ \{H_k, F\}(L) = 0,$$

$$ii)\ \{H_k, H_L\}_R(L) = 0.$$

Proof.

$$i).\{H_k, F\}(L) =< L, [\nabla_{H_k}(L), \nabla_F(L)] >=$$

(using (9))

$$< [L, \nabla_{H_k}(L)], \nabla_F(L) >= 0,$$

since by lemma 4 and by condition of the proposition $[L, \nabla_{H_k}(L)] = 0$.

$$ii).\{H_k, H_l\}_R(L) =$$

$$\frac{1}{2} < L, [R\nabla_{H_k}(L), \nabla_{H_l}(L)] > + \frac{1}{2} < L, [\nabla_{H_k}(L), R\nabla_{H_l}(L)] >=$$

$$-\frac{1}{2} < [L, \nabla_{H_l}(L)], R\nabla_{H_k}(L) > + \frac{1}{2} < [L, \nabla_{H_k}(L)], R\nabla_{H_l}(L) >= 0,$$

by the same argument as above.

Lemma 5 . *Let $H \in \mathcal{F}(P^n)$, $\nabla_H(L) = U = (U_1, \dots, U_n)$ and, for all $i = 1, \dots, n$, we have $[U_i, L_i] = 0$. Then the Hamiltonian system $\frac{\partial F}{\partial t} = \{F, H\}_R$, $F \in \mathcal{F}(P^n)$ is of the form*

$$\frac{\partial L_i}{\partial t} = [(U_i)_+, L_i]$$

on the space P^n.

Proof. Fix $M = (M_1, \dots, M_n) \in P^n$ and take the functional F_M as F. Then

$$\{F_M, H\}_R(L) =$$

$$< L, [\nabla_{F_M}(L), \nabla_H(L)]_R >=< L, [M, U]_R >=$$

$$< L, [M_+, U_+] - [M_-, U_-] >=$$

$$- < [L, U_+], M_+ > + < [L, U_-], M_- >=$$

(using lemma 3 and (9))

$$- < [L, U_+], M_+ > - < [L, U_+], M_- >=$$

(because by condition $[U, L] = 0 \ \ U = U_+ + U_-$)

$$= - < [L, U_+], M > = < [U_+, L], M > .$$

Since $\frac{\partial F_M}{\partial t} = < \frac{\partial L}{\partial t}, M >$ we get what we wanted.

Combining together the lemmas 4 and 5, we get

Proposition 8 . *The Hamiltonian system*

$$\frac{\partial F}{\partial t_m} = \{F, H_m\}_R, \ \ F \in \mathcal{F}(P^n)$$

has the following form on the manifold P' :

$$\frac{\partial L_i}{\partial t_m} = m_i[(L_1 \cdots L_{i-1}^{m_{i-1}} L_i^{m_i - 1} L_{i+1}^{m_{i+1}} \cdots L_n^{m_n})_+, L_i], \ \ i = 1, \ldots, n.$$

Remark 5. This system coincides with the system (6) §3 only if $n = 1$. To present it in a Hamiltonian form, we have to take $n = 2$ and to consider only a part of the Hamiltonians.

Namely, let now $m, k, l \in \mathbf{N}$. We put

$$H_m = \frac{1}{m} \sum_{k+l=m} \binom{m}{k} H_{k,l}.$$

Then from lemma 4 we instantly find

$$\nabla_{H_m}(L_1, L_2) = (\sum_{i+j=m-1; i,j \geq 0} \binom{m-1}{i} L_1^i L_2^j, \sum_{i+j=m-1; i,j \geq 0} \binom{m-1}{j} L_1^i L_2^j)$$

and this expression is of the form (U, U). Taking all together we get the following result.

Proposition 9 . *The Hamiltonian system*

$$\frac{\partial F}{\partial t_m} = \{F, H_m\}_R, \ \ F \in \mathcal{F}(P^2)$$

has the following properties:

i) The variety P' is invariant and on P' the system has the form:

$$\frac{\partial L_1}{\partial t_m} = [(L_1 + L_2)_+^{m-1}, L_1],$$

$$\frac{\partial L_2}{\partial t_m} = [(L_1 + L_2)_+^{m-1}, L_2].$$

ii) The functionals H_m are the integrals of motion and $\{H_m, H_{m'}\}_R = 0$ on P'.

REMARK 6. Let us note that P' is not a Poisson subvariety in P^2. This does not permit to present our dynamical system as a Hamiltonian one on the space P'. However, it is an open question if there exists a Poisson structure on P' such that the property ii) will be preserved and the system would be Hamiltonian.

REMARK 7. The decomposition $P^n = P^n_+ + P^n_-$ used above is not a unique one. We note that the rings P and E admit some non-standard decompositions (as vector spaces) into a direct sum of its subrings. For instance, for $n = 1$ we have, besides $P = P_+ + P_-$, the following decompositions:

$$P = xk[[x]]((\partial^{-1})) + k[x^{-1}]((\partial^{-1})) =$$

$$k[[x]]((\partial^{-1})) + x^{-1}k[x^{-1}]((\partial^{-1})),$$

$$P = xk[[x]](((x\partial)^{-1})) + k[x^{-1}](((x\partial)^{-1})) =$$

$$k[[x]](((x\partial)^{-1})) + x^{-1}k[x^{-1}](((x\partial)^{-1})),$$

It seems that the corresponding Hamiltonian systems have not yet been studied.

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